In a hot, chirally symmetric phase,
π^0 doesn't go into 2γ, but $\pi^0\sigma$ does

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Abstract

In a constituent quark model at nonzero temperature, the amplitude for $\pi^0 \to 2\gamma$ vanishes in a chirally symmetric phase, while that for $\pi^0\sigma \to 2\gamma$ does not.

1. In Memoriam

This paper is dedicated to Tanguy Altherr. It is not the subject on which I spoke at the workshop in his memory, which occured about eight months before I wrote this. In the interim, I think that I've stumbled onto something interesting. I include it here as a gift to Tanguy, as the sort of thing he might have done had he lived.

2. Anomalies

If one constructs the axial current for the fundamental fermion fields of a gauge theory, the divergence of the axial current typically has anomalies: the divergence is not just that given by the classical equations of motion, but has quantum contributions at one loop order.[1] The anomaly is not altered by contributions to higher loop order.[2] Only the ultraviolet region of the one loop integrals contributes to the anomaly, so it is reasonable that the effects of a medium — such as a thermal bath — do not modify the anomaly.[3,4]

The basic point of this note is that while the form of the anomaly in terms of the *fundamental* fields is unrenormalized by a thermal bath, how the anomaly manifests itself in terms of *effective* fields *does* change with temperature. For simplicity I limit myself to nonzero temperature, but the same conclusions hold for a fermi sea. I consider $\pi^0 \to 2\gamma$ as a prototypical process, but similar conclusions hold for $\omega \to \rho\pi$, $\omega \to \pi\pi\pi$, and $\pi\pi \to KKK$. This note is the first announcement of work in progress.[5]

I work in a constituent quark model, with two flavors and three colors of quark fields ψ. I neglect the coupling of quarks to gluons, so the color only contributes a factor of $N_c = 3$ to a quark loop. I include the coupling of quarks to photons, A^μ, and to mesons, Φ. The meson field $\Phi = \sigma t_0 + i\vec{\pi} \cdot \vec{t}$, with σ a 0^+ meson, $\vec{\pi}$ the 0^- pions, and the flavor matrices are $t_0 = \mathbf{1}/2$, $tr(t^a t^b) = \delta^{ab}/2$.

Left and right handed quark fields are constructed by using the projectors $P_{\ell,r} = (1 \mp \gamma^5)/2$, $\psi_{\ell,r} = P_{\ell,r}\psi$; I work in euclidean spacetime with a positive definite metric, and take $(\gamma^5)^2 = 1$. A chirally symmetric lagrangian is

$$\mathcal{L} = \overline{\psi}_\ell \, \slashed{D} \, \psi_\ell + \overline{\psi}_r \, \slashed{D}\psi_r + 2\widetilde{g}\left(\overline{\psi}_\ell \Phi\psi_r + \overline{\psi}_r \Phi^\dagger\psi_\ell\right) . \tag{1}$$

$\slashed{D} = (\slashed{\partial} - iq \, \slashed{A})$, where q is a matrix for the electric charge of the up and down quarks, $q = e(t_3 + t_0/3)$. Excluding the electromagnetic coupling, this lagrangian is manifestly invariant under global $SU_\ell(2) \times SU_r(2)$ chiral rotations Ω_ℓ and Ω_r,

$$\psi_{\ell,r} \to \Omega^\dagger_{\ell,r}\psi_{\ell,r} \quad , \quad \Phi \to \Omega^\dagger_\ell \Phi \, \Omega_r . \tag{2}$$

Explicitly,
$$\mathcal{L} = \overline{\psi}\left(\slashed{D} + 2\widetilde{g}\left(\sigma t_0 + i\vec{\pi} \cdot \vec{t}\gamma^5\right)\right)\psi . \tag{3}$$

I do not consider the dynamics of either the scalar or quark fields. All I will do is to derive the effective lagrangian between the scalar and photon fields which is induced by integrating out the quarks at one loop order.

I assume that chiral symmetry breaking occurs at zero temperature, $\langle\sigma\rangle = \sigma_0$, so the consituent quark mass is $m = \widetilde{g}\sigma_0$. In a sigma model with two flavors, $\sigma_0 = f_\pi$, where $f_\pi = 93\,MeV$ is the pion decay constant.

The Feynman rules required are the following: the quark propagator is $1/(i\,\slashed{P}+m)$, the coupling between σ and a quark line is $-\widetilde{g}$, the coupling between a pion π^a and a quark line is $-2i\widetilde{g}t^a\gamma^5$, and the coupling between a photon and a quark line is $+iq\gamma^\mu$.

3. $\pi^0 \to 2\gamma$ at zero temperature

In this section I calculate $\pi^0 \to 2\gamma$ in a constituent quark model.[1,6] Let the two photons be $A^\mu(P_1)$ and $A^\nu(P_2)$, where P_1 and P_2 are the four momenta. There are two triangle diagrams which contribute; one is

$$-\frac{i\widetilde{g}e^2 N_c}{3} tr_K \, tr_{Dirac}\left(\gamma^5 \frac{1}{(i(\slashed{K}+\slashed{P}_1)+m)} \gamma^\mu \frac{1}{(i\,\slashed{K}+m)} \gamma^\nu \frac{1}{(i(\slashed{K}-\slashed{P}_2)+m)}\right) . \tag{4}$$

$tr_K = \int d^4K/(2\pi)^4$ is the integral over the loop momenta K, and tr_{Dirac} the Dirac trace. The latter is done using the identity

$$tr\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = 4\,\epsilon^{\mu\nu\alpha\beta} , \tag{5}$$

with $\epsilon^{\mu\nu\alpha\beta}$ the antisymmetric tensor. Doing the Dirac algebra, this diagram becomes

$$-\frac{4i\tilde{g}e^2 N_c}{3} m \, \epsilon^{\mu\nu\alpha\beta} P_1^\alpha P_2^\beta \, I(P_1, P_2, m) \,, \tag{6}$$

where $I(P_1, P_2, m)$ is the loop integral

$$I(P_1, P_2, m) = tr_K \, \frac{1}{(K^2 + m^2)((K + P_1)^2 + m^2)((K - P_2)^2 + m^2)} \,. \tag{7}$$

This integral is finite and well defined. In the limit of small momenta, the dependence on P_1 and P_2 in the integral can be neglected, with

$$I(0, 0, m) = tr_K \, \frac{1}{(K^2 + m^2)^3} = \frac{1}{32\pi^2 m^2} \,. \tag{8}$$

The second diagram, which follows by interchanging P_1 and P_2, and μ and ν, contributes equally. Putting all of this together, the effective lagrangian for $\pi^0 \to 2\gamma$ is

$$\mathcal{L}_{\pi^0 \to 2\gamma} = +i \, \frac{e^2 N_c}{96\pi^2 f_\pi} \, \pi^0 \, \epsilon^{\mu\nu\alpha\beta} \, F^{\mu\nu} \, F^{\alpha\beta} \,. \tag{9}$$

To derive this, I have used the relation $m = \tilde{g} f_\pi$. To anticipate the results in the following sections, I note that after integration by parts,

$$\mathcal{L}_{\pi^0 \to 2\gamma} = -i \, \frac{e^2 N_c}{24\pi^2 f_\pi^2} \, \left(f_\pi \partial^\mu \pi^0 \right) \, \epsilon^{\mu\nu\alpha\beta} \, A^\nu \, \partial^\alpha \, A^\beta \,. \tag{10}$$

Given the derivation, it is clear that this is only the leading term in an expansion in low momentum. The integral $I(P_1, P_2, m)$ can be expanded in powers of the external momenta, in powers of $P_1^2/m^2, P_2^2/m^2$, and so on. In an effective lagrangian these terms would becomes powers of ∂^2/m^2 acting upon the various fields. Thus at least for the couplings between pions and photons, there is no such thing as "the" anomaly. The lowest term, $\mathcal{L}_{\pi^0 \to 2\gamma}$, *is* unique in that it is part of the Wess-Zumino-Witten term, whose form can be derived from very general considerations,[7] apart from an overall constant proportional to the number of colors, N_c. But even just from the integral at one loop order, there will still be an infinity of couplings between π^0 and $\epsilon^{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta}$; because of the presence of the antisymmetric tensor, these couplings can be uniquely identified with the anomaly.

4. $\pi^0 \to 2\gamma$ at non zero temperature

Now consider the calculation of $\pi^0 \to 2\gamma$ in a thermal bath, at a nonzero temperature T. This is simply a matter of computing the integral $I(P_1, P_2, m)$ at nonzero temperature. I assume that the theory is in a phase near a point at which chiral symmetry is restored, so that the constituent quark mass m is much less than the

temperature. Thus m can be neglected in the integral, as can P_1 and P_2. The integral is that for a fermion loop, so the timelike component of the momentum $k^0 = (2n + 1)\pi T$, with a sum over all integers n. To do the integral it is easiest to integrate over the spatial momentum k first and then do the sum over n:

$$tr_K \frac{1}{(K^2)^3} = T \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2 + (k_0)^2)^3}$$

$$= \frac{1}{16\pi^4 T^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} = \frac{7\zeta(3)}{128\pi^4 T^2} . \tag{11}$$

$\zeta(3) = \sum_{n=1}^{\infty} 1/n^3 = 1.20206...$ is a type of zeta function, $\zeta(r) = \sum_{n=1}^{\infty} 1/n^r$.

In the limit that $m \ll T$, at nonzero temperature the effective lagrangian for $\pi \rightarrow 2\gamma$ is then equal to

$$\mathcal{L}_{\pi^0 \rightarrow 2\gamma}(T \neq 0) = -i \frac{7\zeta(3)e^2\tilde{g}^2 N_c}{96\pi^4 T^2} \left(\sigma_0 \partial^\mu \pi^0\right) \epsilon^{\mu\nu\alpha\beta} A^\nu \partial^\alpha A^\beta . \tag{12}$$

This demonstrates that in a chirally symmetric phase, where $\sigma_0 = 0$, the amplitude for $\pi^0 \rightarrow 2\gamma$ *vanishes*! Actually, we could easily have anticipated that, contrary to previous authors,[8] the form of the coupling between $\pi^0 \rightarrow 2\gamma$ *must* change at nonzero temperature. At zero temperature, this amplitude is proportional to $1/f_\pi$. Assume, for the purposes of argument, that the restoration is a second order phase transition, at which f_π vanishes smoothly. If the form of this coupling didn't change, it would diverge, which is nonsensical for a physical amplitude.

This doesn't explain why the amplitude for $\pi \rightarrow 2\gamma$ *vanishes* in a chirally symmetric phase. I have written the above expression in a suggestive form. The coupling between the Φ field and photons must be gauge invariant; as a term induced by the anomaly, it should also involve the antisymmetric tensor. With the total sigma field equal to $\sigma_0 + \sigma$, the contribution of the scalar fields to the axial current is

$$\vec{J}_{axial}^\mu = (\sigma_0 + \sigma)\partial^\mu \vec{\pi} - \vec{\pi}\partial^\mu \sigma . \tag{13}$$

Thus the expression for $\mathcal{L}_{\pi^0 \rightarrow 2\gamma}(T \neq 0)$ could simply be the first term in the expression

$$\mathcal{L}_{\pi^0 \sigma \rightarrow 2\gamma}(T \neq 0) = -i \frac{7\zeta(3)e^2\tilde{g}^2 N_c}{96\pi^4 T^2} J_{3,axial}^\mu \epsilon^{\mu\nu\alpha\beta} A^\nu \partial^\alpha A^\beta , \tag{14}$$

$J_{3,axial}^\mu = 2 \, tr(t^3 \vec{J}_{axial}^\mu)$.

If correct, Eq. (14) predicts that while the amplitude for $\pi^0 \rightarrow 2\gamma$ vanishes in a chirally symmetric phase, that for $\pi^0 \sigma \rightarrow 2\gamma$ does *not*. It is relatively easy to check this. In all, there are six diagrams which contribute. One diagram is

$$\frac{ie^2\tilde{g}^2 N_c}{3} tr_K \, tr_{Dirac} \left(\frac{\gamma^5 \not{K} \gamma^\mu (\not{K} - \not{P}_1) \gamma^\nu (\not{K} - \not{P}_1 - \not{P}_2)(\not{K} + \not{P}_4)}{K^2 (K - P_1)^2 (K + P_4)^2 (K - P_1 - P_2)^2}\right) . \tag{15}$$

As before, the two photons are $A^\mu(P_1)$ and $A^\nu(P_2)$; the momentum of the σ is P_3, that of the pion P_4. For arbitrary momenta the Dirac trace in this expression involves trace of γ^5 times six γ's, which is involved. To simplify the Dirac algebra I assume that $P_4 = 0$. (I checked that the result obtained for $P_3 = 0$ has the proper change in sign as predicted from Eq. (14).) After doing the Dirac algebra, the diagram becomes

$$\frac{-4ie^2\widetilde{g}^2 N_c}{3} \, tr_K \left(\frac{\epsilon^{\mu\nu\alpha\beta} K^\alpha P_2^\beta}{K^2 (K+P_1)^2 (K-P_2)^2} \right) . \tag{16}$$

The denominators can be expanded in small momenta, such as

$$\frac{1}{(K+P_1)^2} = \frac{1}{K^2} - \frac{2K \cdot P_1}{(K^2)^2} + \dots . \tag{17}$$

Remember that at nonzero temperature, the fermion loop momenta k^0 is an odd multiple of πT, while the bosonic momenta p^0 are even multiples of πT. In order to be able to expand in powers of the external momenta, then, implicitly it is necessary to assume that $p^0 = 0$ for each of the external momenta. Consequently, one of the indices μ or ν must be time like.

The integral

$$tr_K \frac{K^\alpha K^\beta}{(K^2)^4} = \left(\delta^{\alpha\beta} + 2n^\alpha n^\beta \right) \frac{7\zeta(3)}{128\pi^4 T^2} \tag{18}$$

is required, where $n^\alpha = \delta^{\alpha 0}$. Actually, the term proportional to n^α doesn't contribute, because of the assumption that $n \cdot P_1 = n \cdot P_2 = 0$. The diagram then equals

$$+ \ i \ \frac{7\zeta(3)e^2\widetilde{g}^2 N_c}{288\pi^4 T^2} \, \epsilon^{\mu\nu\alpha\beta} \, P_1^\alpha \, P_2^\beta . \tag{19}$$

This is only one of six diagrams. All diagrams contribute equally, so in all the amplitude is six times this expression. This agrees with Eq. (14), after remembering that because there are two A^μ's, there is a factor of two which arises in going from the effective lagrangian to the amplitude.

A general understanding of how anomalous interactions induced by fermion loops manifest themselves in a chirally symmetric phase at nonzero temperature can be guessed from the results for $\pi^0 \to 2\gamma$. I define an anomalous interaction as one which involves the antisymmetric tensor, $\epsilon^{\mu\nu\alpha\beta}$. This is then contracted with (gauge invariant) functions of the external gauge fields and the scalar field, Φ. Rather obviously, in a chirally symmetric phase the scalar field Φ must enter in a chirally invariant manner. In general, the operators so constructed have a mass dimension greater than four. At zero temperature, the proper mass dimension is provided by adding powers of $1/f_\pi$; in a chirally symmetric phase at nonzero temperature, powers of $1/T$ enter. Lastly, the expression must be invariant under parity, under which $\Phi \leftrightarrow \Phi^\dagger$.

For example, the axial current is the difference of left and right handed currents,

$$J_{3,axial}^\mu = i \, tr \left(t^3 \left(\Phi \partial^\mu \Phi^\dagger - \partial^\mu \Phi \Phi^\dagger + \partial^\mu \Phi^\dagger \Phi - \Phi^\dagger \partial^\mu \Phi \right) \right) . \tag{20}$$

The reason why $\pi^0 \to 2\gamma$ vanishes in a chirally symmetric phase, then, is simply because there is no chirally invariant function which is linear in the pion field. The simplest chirally invariant function is bilinear in Φ, so the interaction begins as $\pi^0 \sigma \to 2\gamma$.

Similar conclusions hold for other anomalous interactions. All interactions of the ω meson are anomalous: at zero temperature the dominant couplings of the ω are

$$\frac{1}{f_\pi} \, \epsilon^{\mu\nu\alpha\beta} \, \omega^\mu \, \partial^\nu \vec{\rho}^\alpha \cdot \partial^\beta \vec{\pi} \; , \tag{21}$$

which governs $\omega \to \rho\pi$ and

$$\frac{1}{f_\pi^3} \, \epsilon^{\mu\nu\alpha\beta} \, \omega^\mu \, \partial^\nu \vec{\pi} \cdot \left(\partial^\alpha \vec{\pi} \times \partial^\beta \vec{\pi} \right) \; , \tag{22}$$

which describes $\omega \to \pi\pi\pi$.

Using the rules described above, in a chirally symmetric phase at nonzero temperature the analogous interactions are

$$\frac{1}{T^2} \, \epsilon^{\mu\nu\alpha\beta} \, \omega^\mu \, \partial^\nu \vec{\rho}^\alpha \cdot \vec{J}_{axial}^\beta \tag{23}$$

and

$$\frac{1}{T^4} \, \epsilon^{\mu\nu\alpha\beta} \, \omega^\mu \, tr \left(\Phi^\dagger \, \partial^\nu \Phi \, \partial^\alpha \Phi^\dagger \, \partial^\beta \Phi \; - \; \Phi \, \partial^\nu \Phi^\dagger \, \partial^\alpha \Phi \, \partial^\beta \Phi^\dagger \right) \; . \tag{24}$$

There are dimensionless coefficients in front of each expression, which involve the various coupling constants and pure numbers, such as $\zeta(3)$ and $\zeta(5)$, respectively. The form of the second expression is dictated by parity invariance, as the Φ dependent term must be odd under $\Phi \leftrightarrow \Phi^\dagger$. From the form of these expressions, it is evident that $\omega \to \rho\pi$ and $\omega \to \pi\pi\pi$ vanish in a chirally symmetric phase, but that the processes $\omega \to \rho\pi\sigma$ and $\omega \to \pi\pi\pi\sigma$ do not. Under the standard assumption of vector meson dominance,[9] this suggests that the width of the ω does not significantly increase with temperature.

The generalization of the Wess-Zumino-Witten term to nonzero temperature can also be derived by these arguments. At zero temperature, the Wess-Zumino-Witten term governs processes such as $KK \to \pi\pi\pi$. In a chirally symmetric phase at nonzero temperature, $KK \to \pi\pi\pi$ vanishes, but $KK \to \pi\pi\pi\sigma$ is allowed. The extension of the Wess-Zumino-Witten term is not a simple generalization of that at zero temperature, as has been previously conjectured.[10]

5. References

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